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HEAT CONDUCTIVITY OF THE ADJOINING PLATES WITH A PLANE HEAT SOURCE BETWEEN THEM

I. M. Fedotkin, E. V. Verlan,
I. D. Chebotaresku, and S. V. Evtukhovich

UDC 536.2:643.343.320.191.8

The heat conduction problem in a two-layered plate with a plane heat source between the layers is solved by the introduction of an unknown heat flux determined later from a Volterra integral equation of the second kind by the Bubnov–Galerkin method.

The Laplace transform method has limited application in solving heat conduction problems in two and multilayered walls because of the complexity of executing the inversion of the transform. A method is known for the reduction of such problems to the solution of Volterra type integral equations of the second kind in the unknown heat flux on the junction between the walls [1-3].

However, as is shown in [3], finite integral transforms result in a solution in the form of infinite poorly convergent series requiring the application of special methods to improve their convergence. Moreover, representation of the kernels of the Volterra integral equations in the form of infinite series does not permit obtaining the exact solution of the problem in analytic form.

It is expedient to use approximate methods based on the combined utilization of the Laplace integral transform and the Ritz or Bubnov–Galerkin method to solve heat conduction problems in multilayered walls. Such a method is developed in [4] for bodies of the simplest shape. In this case the solution of the Volterra integral equations relative to interlayered thermal fluxes, and therefore the solution of the problem is also successfully obtained in analytic form since the kernels of the integral equations consist of the simplest analytic functions without series.

It is shown in [4] that numerical values of the temperature fields obtained by using approximate and exact solutions agree with high accuracy.

The heat conduction problem considered in this paper is that a plane heat source, independent of the coordinates and time, acts between two infinite plates starting from a certain time. The heat transfer at the outer surfaces of the plates occurs according to the Newton law for a constant heat transfer coefficient. The thermophysical characteristics of the plates are independent of the temperature. The temperature of the plates at the

50th Anniversary of the October Revolution, Kiev Polytechnic Institute. S. Lazo Kishinev Polytechnic Institute. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 45, No. 3, pp. 493-498, September, 1983. Original article submitted March 18, 1982.

initial instant T_0 is constant in the coordinate and equals the temperature of the surrounding medium T_m. The origin is on the outer surface of the left wall such that $0 \le x = \delta_2$, $\delta_2 = \delta_1 + (\delta_2 - \delta_1)$, δ_1 is the thickness of the left, and $\delta_2 - \delta_1$ is the thickness of the right wall.

The problem is written as follows in dimensionless form:

$$\frac{\partial \Theta_i}{\partial \mathrm{Fo}} = K_{a_i} \frac{\partial^2 \Theta_i}{\partial X^2}, \ i = 1, 2, \tag{1}$$

$$\mathrm{Fo} = 0; \ \Theta_1(X, 0) = \Theta_2(X, 0) = 0, \tag{2}$$

$$X = 0; \ \frac{\partial \Theta_1}{\partial X} = \mathrm{Bi}_1 \Theta_1, \tag{3}$$

$$X = X_1; \ \Theta_1 = \Theta_2, \ \frac{\partial \Theta_1}{\partial X} - K_\lambda \frac{\partial \Theta_2}{\partial X} = \mathrm{Ki}_0, \tag{4}$$

$$X = X_2; \ -\frac{\partial \Theta_2}{\partial X} = \mathrm{Bi}_2 \Theta_2, \tag{5}$$

where $0 \le X \le X_2$, $0 \le \mathrm{Fo} \le \pm\infty$, K_{a_i}, K_λ, Ki_0, $\mathrm{Bi}_{1,2}$ are constants.

Because $\mathrm{Ki}_0 = \mathrm{const}$ the problem (1)-(5) formulated can be separated into two simpler problems. Let us use the notation $\Theta_i = u_i + v_i$, where u_i is the stationary component and v_i the nonstationary component of the temperature field. Then on the basis of the principle of superposition of fields, we have

$$\begin{gathered} \frac{\partial^2 u_j}{\partial X^2} = 0, \ j = 1; \ 2, \ \left.\frac{\partial u_1}{\partial X^2}\right|_{X=0} = \mathrm{Bi}_1 u_1, \ u_1|_{X=X_1} = u_2|_{X=X_1}, \\ -\left.\frac{\partial u_2}{\partial X}\right|_{X=X_2} = \mathrm{Bi}_2 u_2, \ \left.\frac{\partial u_1}{\partial X}\right| - K_\lambda \left.\frac{\partial u_2}{\partial X}\right|_{X=X_1} = \mathrm{Ki}_0. \end{gathered} \tag{6}$$

The solution of the problem (6) is the following: $u_j = c_{1j}X + c_{2j}$, $c_{1j} = \mathrm{Ki}_0 K_{1j}[1 + \mathrm{Bi}_2(X_2 - X_1)]/\Delta$, $K_{11} = \mathrm{Bi}_1$, $K_{12} = 1$, $c_{2j} = \mathrm{Ki}_0 K_{2j}(1 + X_1\mathrm{Bi}_1)/\Delta$, $K_{21} = -\mathrm{Bi}_2$, $K_{22} = 1 + X_2\mathrm{Bi}_2$, $\Delta = K_\lambda \mathrm{Bi}_2(1 + X_1\mathrm{Bi}_1) + \mathrm{Bi}_1[1 + \mathrm{Bi}_2(X_2 - X_1)]$. The nonstationary part of the temperature field is described by the system of equations:

$$\frac{\partial v_i}{\partial \mathrm{Fo}} - K_{a_i} \frac{\partial^2 v_i}{\partial X^2} = 0, \ i = 1; \ 2, \tag{7}$$

$$\left.\frac{\partial v_1}{\partial X}\right|_{X=0} = \mathrm{Bi}_1 v_1, \ (a) \ v_1|_{X=X_1} = v_2|_{X=X_1}, \ (c) \tag{8}$$

$$-\left.\frac{\partial v_2}{\partial X}\right|_{X=X_2} = \mathrm{Bi}_2 v_2, \ (b) \ \left.\frac{\partial v_1}{\partial X}\right|_{X=X_1} - K_\lambda \left.\frac{\partial v_2}{\partial X}\right|_{X=X_1} = 0, \ (d)$$

$$v_1(X, 0) = -u_1(X), \ v_2(X, 0) = -u_2(X). \tag{9}$$

The potentials v_1 and v_2 are interrelated by means of the boundary conditions (8). Introducing the unknown heat flux function Ki(Fo) on the junction between the plates [3] we separate the problem (7)-(9) into two independent problems for v_1 and v_2. We here write the conditions for equality of the heat flux (8d) between the plates in the form of two equations

$$\left.\frac{\partial v_1}{\partial X}\right|_{X=X_1} = \mathrm{Ki}(\mathrm{Fo}), \tag{10}$$

$$\left.\frac{\partial v_2}{\partial X}\right|_{X=X_1} = \frac{1}{K_\lambda} \mathrm{Ki}(\mathrm{Fo}). \tag{11}$$

After such a separation, the potentials v_1 and v_2 remain interrelated by condition (8c).

Applying the Laplace transform to (7)-(10), we obtain for the left plate

$$\frac{\partial^2 \bar{v}_1}{\partial X^2} - p\bar{v}_1 - F_1(X) = 0, \tag{12}$$

$$\left.\frac{\partial v_1}{\partial X}\right|_{X=0} = \mathrm{Bi}_1 v_1; \quad \left.\frac{\partial \bar{v}_1}{\partial X}\right|_{X=X_1} = \overline{\mathrm{Ki}}\,(p), \tag{13}$$

where $F_1(X) = c_{11}X + c_{12}$.

We seek the solution of the problem (12)-(13) in the family of functions of the form [4]:

$$\bar{v}_1(X,\ p) = \left(\frac{1}{\mathrm{Bi}_1} + X\right)\overline{\mathrm{Ki}}\,(p) + \sum_{k=1}^{n} \bar{a}_k(p)\,\psi_k(X), \tag{14}$$

where $\psi_k(X)$ is a certain system of coordinate functions satisfying the homogeneous boundary conditions (13). Such functions for the problem (12)-(13) are

$$\psi_k(X) = \left(\frac{1}{2}X^2 - X_1 X - \frac{X_1}{\mathrm{Bi}_1}\right) X^{2(k-1)}. \tag{15}$$

For a given selection of functions ψ_k the solution (14) satisfies the boundary conditions (13) exactly, and the differential equation (12) approximately, whereupon the following residual is obtained:

$$\varepsilon_k\,[\bar{a}_1(p),\ \ldots,\ a_k(p),\ X] = \frac{\partial^2 \bar{v}_1}{\partial X} - p\bar{v}_1 - F_1(X) \neq 0.$$

Let us demand the residual be orthogonal to the coordinate functions $\psi_j(X)$:

$$\int_0^{X_1} \varepsilon_k\,[\bar{a}_1(p),\ \bar{a}_2(p),\ \ldots,\ \bar{a}_k(p),\ X]\,\psi_j(X)\,dX = 0. \tag{16}$$

Solving (16), we obtain a system of algebraic equations to determine the coefficients $\bar{a}_k(p)$:

$$\sum_{k=1}^{n} \bar{a}_k(p)\,(A_{jk}^{(1)} - pB_{jk}^{(1)}) = \bar{D}_j^{(1)}(p), \tag{17}$$

where

$$A_{jk}^{(1)} = \int_0^{X_1} \frac{\partial^2 \psi_k(X)}{\partial X^2}\,\psi_j(X)\,dX; \quad B_{jk}^{(1)} = \int_0^{X_1} \psi_k(X)\,\varphi_j(X)\,dX;$$

$$\bar{D}_j^{(1)}(p) = p\overline{\mathrm{Ki}}\,(p)\int_0^{X_1}\left(\frac{1}{\mathrm{Bi}_1} + X\right)\psi_j(X)\,dX + \int_0^{X_1} F_1(X)\,\psi_j(X)\,dX = p\,\overline{\mathrm{Ki}}\,(p)\,D_{j1}^{(1)} + D_{j2}^{(1)}.$$

Setting n = 1, 2, or 3 in (17), we obtain truncated systems of Bubnov–Galerkin equations of the first, second, or third order, from which we determine the coefficients $\bar{a}_k(p)$. In engineering computations sufficient accuracy for practical computations is achieved in solving truncated systems of first- or second-order equations (17).

Solving the truncated first-order system (17) and applying the inverse Laplace transform to $\bar{a}_k(p)$, we find the solution of the problem for the left plate

$$v_1(X,\ \mathrm{Fo}) = \left[\frac{1}{\mathrm{Bi}_1} + X + A_1\psi_1(X)\right]\mathrm{Ki}\,(\mathrm{Fo}) - A_1R_1\psi_1(X)\times$$
$$\times\int_0^{\mathrm{Fo}} e^{-R_1(\mathrm{Fo}-\theta)}\,\mathrm{Ki}\,(\theta)\,d\theta + B_1\psi_1(X)\exp(-R_1\mathrm{Fo}), \tag{18}$$

where

$$A_1 = \frac{1}{B_{11}^{(1)}}\left[\frac{5}{6\mathrm{Bi}_1}X_1^3 + \frac{1}{\mathrm{Bi}_1}X_1^2 + \frac{5}{24}X_1^4\right];$$

$$B_{11}^{(1)} = \frac{2}{15}X_1^5 + \frac{2}{3\mathrm{Bi}_1}X_1^4 + \frac{1}{\mathrm{Bi}_1^2}X_1^3,$$

$$B_1 = \frac{1}{B_{11}^{(1)}} \left[\frac{5}{24} c_{11} X_1^4 + \left(\frac{c_{11}}{2\mathrm{Bi}_1} + \frac{c_{12}}{3} \right) X_1^3 + \frac{c_{12}}{\mathrm{Bi}_1} \right];$$

$$R_1 = \frac{1}{B_{11}^{(1)}} \left(\frac{1}{3} X_1^3 + \frac{1}{\mathrm{Bi}_1} X_1^2 \right);$$

$$\psi_1(X) = \frac{1}{2} X^2 - X_1 X - X_1/\mathrm{Bi}_1.$$

We obtain the solution for the right plate in an analogous manner:

$$v_2(X, \mathrm{Fo}) = \frac{1}{K_\lambda} \left[X - X_2 - \frac{1}{\mathrm{Bi}_2} + K_\lambda \varphi_1(X) \right] - A_2 R_2 \varphi_1(X) \times$$
$$\times \int_0^{\mathrm{Fo}} e^{-R_2(\mathrm{Fo}-\theta)} \mathrm{Ki}(\theta)\, d\theta + B_2 \varphi_1(X) \exp(-R_2 \mathrm{Fo}), \tag{19}$$

where

$$A_2 = \frac{1 + \mathrm{Bi}_2 X_2}{B_{11}^{(2)} K_\lambda K_a^2 \mathrm{Bi}_2} \left[\frac{1}{6} (X_2^3 - X_1^3) - \frac{X_1}{2} (X_2^2 - X_1^2) - \right.$$
$$\left. - \frac{N}{\mathrm{Bi}_2} (X_2 - X_1) \right] - \frac{1}{B_{11}^{(2)} K_\lambda K_a^2} \left[\frac{1}{8} (X_2^4 - X_1^4) - \frac{X_1}{3} (X_2^3 - X_1^3) - \frac{N}{2\mathrm{Bi}_2} (X_2^2 - X_1^2) \right];$$

$$B_2 = \frac{c_{21}}{B_{11}^{(2)}} \left[-\frac{1}{8} (X_2^4 - X_1^4) + \frac{X_1}{3} (X_2^3 - X_1^3) + \frac{N}{2\mathrm{Bi}_2} (X_2^2 - X_1^2) \right] +$$
$$+ \frac{c_{22}}{B_{11}^{(2)}} \left[-\frac{1}{6} (X_2^3 - X_1^3) + \frac{X_1}{2} (X_2^2 - X_1^2) + \frac{N}{\mathrm{Bi}_2} (X_2 - X_1) \right];$$

$$R_2 = \frac{1}{B_{11}^{(2)}} \left[-\frac{1}{6} (X_2^3 - X_1^3) + \frac{X_1}{2} (X_2^2 - X_1^2) + \frac{N}{2\mathrm{Bi}_2} (X_2 - X_1) \right];$$

$$B_{11}^{(2)} = \frac{1}{K_a^2} \left[\frac{X_2^5 - X_1^5}{20} + \left(X_1^2 - \frac{N}{\mathrm{Bi}_2} \right) \frac{X_2^3 - X_1^3}{3} + \frac{2NX_1}{\mathrm{Bi}_2} \frac{X_2^2 - X_1^2}{2} + \frac{N^2}{\mathrm{Bi}_2^2} (X_2 - X_1) \right];$$

$$N = X_2 - X_1 + \frac{\mathrm{Bi}_2}{2} X_2 - X_1 X_2 \mathrm{Bi}_2;$$

$$\varphi_j(X) = \left(\frac{1}{2} X^2 - X_1 X - \frac{N}{\mathrm{Bi}_2} \right) X^{2(j-1)}.$$

The unknown heat flux Ki(Fo) is determined by substituting (18) and (19) into (20), whereupon a Volterra integral equation of the second kind in Ki(Fo) is obtained. Because of the simplicity of the functions in (18) and (19), the integral equation can be solved analytcially by an operator method.

The solution has the form

$$\mathrm{Ki}(\mathrm{Fo}) = \sum_{j=1}^{2} (-1)^j \Phi_{2j} \exp[\nu_j(X) \mathrm{Fo}], \tag{20}$$

where

$$\Phi_{2j} = \frac{\Phi_{11}(X) \nu_j(X) - \Phi_{12}(X)}{\nu_2(X) - \nu_1(X)},$$

$$\Phi_{11}(X) = \frac{1}{M(X)} [B_2 \varphi_1(X) - B_1 \psi_1(X)],$$

$$\Phi_{12} = \frac{1}{M(X)} [B_1 \psi_1(X) R_2 - B_2 \varphi_1(X) R_1];$$

$$M(X) = \frac{1}{\mathrm{Bi}_1} + X + A_1 \psi_1(X) + \frac{X_2 - X}{K_\lambda} + \frac{1}{K_\lambda \mathrm{Bi}_2} - \varphi_1(X), \ \nu(X) -$$

are roots of the equations

$$M(X)p^2 + B(X)p + c(X) = 0,$$
$$B(X) = M(X)(R_1 + R_2) - A_1R_1\psi_1(X) + A_2R_2\psi_1(X),$$
$$c(X) = R_1R_2M(X) - A_1R_1R_2\psi_1(X) + A_2R_1R_2\varphi_1(X).$$

The roots $\nu_j(X)$ correspond to the physical meaning of the problem if they are real and negative.

Substituting (20) into (18) and (19), we find the explicit form of the functions $v_i(X, Fo)$, $i = 1, 2$, and taking account of the expression $\Theta_i = u_i + v_i$ the solution of the problem in final form becomes

$$\Theta_i = c_{i1}X + c_{i2} + \sum_{j=1}^{2} (-1)^j W_{ij}(X) e^{\nu_j(X)\mathrm{Fo}} - W_{i3}(X) e^{-R_i \mathrm{Fo}}, \tag{21}$$

where

$$W_{ij} = \varkappa_i(X)\ \Phi_{2(j/2)} - \frac{A_iR_i\beta_i(X)\Phi_{2(j/2)}(X)}{R_i - \nu_i(X)},$$

$$W_{i3}(X) = A_iR_i\beta(X) \sum_{j=1}^{2} (-1)^{j+1} \frac{\Phi_{2(j/2)}(X)}{R_i + \nu_j(X)} - B_i\beta_i(X),\ i = j = 1;\ 2,$$

$$\varkappa_1(X) = \frac{1}{\mathrm{Bi}_1} + X + A_1\psi_1(X),$$

$$\varkappa_2(X) = \left[X - X_2 - \frac{1}{\mathrm{Bi}_2} + K_\lambda\varphi_1(X)\right]/K_\lambda.$$

The approximate solution (21) of the problem (1)-(5) permits computation of the temperature field in plates with not more than 5% error for $\mathrm{Fo} \geq 0.05$ [4]. In contrast to the solution of problem (1)-(5) obtained in [3], the solution (21) permits computation of the temperature field in a two-layered plate without involving numerical methods to determine the function Ki(Fo).

NOTATION

$K_{a_1} = a_1/a_1 = 1$, $K_{a_2} = a_2/a_1$, $K_\lambda = \lambda_2/\lambda_1$, $\mathrm{Bi} = \alpha\delta_2/\lambda_1$, Biot criterion; $\mathrm{Ki}_0 = q_0\delta_2/\lambda_1(T_M - T_c)$, Kirpichev criterion; $\mathrm{Fo} = \tau a_1/\delta_2^2$, Fourier criterion; $\Theta = (T - T_0)/(T_M - T_0)$, dimensionless temperature; $T_0 = T_S$, temperature of the surrounding medium; T_m, maximum temperature of the material; q_0, intensity of the plane heat source; and $X = x/\delta_2$, $X_1 = \delta_1/\delta_2$, $X_2 = \delta_2/\delta_2 = 1$.

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